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A COLOURED WINDOW ON PRE-SERVICE TEACHERS' CONCEPTIONS OF RATIONAL NUMBERS

ABSTRACT. In undergraduate mathematics courses, pre-service elementary school teachers are often faced with the task of re-learning some of the concepts they themselves struggled with in their own schooling. This often involves different cognitive processes and psychological issues than initial learning: pre-service teachers have had many more opportunities to construct understandings and representations than initial learners, some of which may be more complex and engrained; pre-service teachers are likely to have created deeply-held–and often negative–beliefs and attitudes toward certain mathematical ideas and processes. In our recent research, we found that pre-service teachers who used a particular computer-based microworld, one emphasising visual representations of and experimental interactions with elementary number theory concepts, overcame many cognitive and psychological difficulties reported in the literature. In this study, we investigate the possibilities of using a similarly-designed microworld that involves a set of rational number concepts. We describe the affordances of this microworld, both in terms of pre-service teacher learning and research on pre-service teacher learning, namely, the helpful "window" it gave us on the mathematical meaning-making of pre-service teachers. We also show how their interactions with this microworld provided many with a new and aesthetically-rich set of visualisations and experiences.

1. INTRODUCTION

In many undergraduate mathematics courses, pre-service elementary school teachers are faced with the task of re-learning many of the concepts they themselves struggled with in their own schooling. They are expected to develop a profound mathematical background in a subject that, for many, makes them anxious and, for most, poses serious conceptual difficulties. It seems clearly misguided to ask these pre-service elementary teachers to learn the concepts they will have to teach – and which they themselves had difficulty learning – in the same way these concepts have previously been encountered. First, from a psychological point of view: only strong, positive mathematical experiences can help learners overcome existing fears and anxieties (Goldin, 2000). Second, from a cognitive point of view: better understanding, according to Skemp (1971) depends on the construction of richer, more multi-dimensional schemas, in which previous understanding can be assimilated. As Piaget's theory maintains, the construction of richer schemas can only occur through a process of dis-equilibration caused by a perceived problem or unexpectedness and subsequent re-equilibration through assimilation into existing schemas or, if necessary, reconstruction of new schemas.

In this paper, we discuss the re-learning of rational numbers (particularly those often referred to as 'fractions' and 'decimals' in the context of school mathematics) by pre-service elementary teachers. We present a computer-based microworld that has been found to provide an unexpected and compelling way of re-presenting these frequently abhorred mathematical objects, and to encourage and support mathematical exploration and problem solving (Sinclair, 2001). Our goal in using this microworld was to gain a window on pre-service teachers' understanding of certain concepts involving rational numbers and to explore the benefits and drawbacks of using such a microworld in pre-service courses.

2. THEORETICAL CONSIDERATIONS

Learning is about acquiring knowledge that can be a piece of information, a skill, an understanding, or even a perspective. Re-learning for prospective elementary teachers is not simply adding knowledge to what has previously been incorporated into a learner's repertoire. It is revisiting and reconstructing conceptions that may have been developed long ago, and been affected by myriad – and often forgotten – experiences both in and out of the mathematics classroom. It is often said that classroom teachers must develop a better understanding of the kinds of mathematical conceptions that their students *already* hold, before coming into the classroom. The same must be said of teacher educators, who are frequently faced with the less malleable conceptions of pre-service teachers, who have developed and reinforced their conceptions over a much broader set of everyday as well as classroom experiences. Since prior structures have been constructed in a learner's mind some time ago, the reconstruction and reorganisation processes involved in re-learning are more challenging for the learner as well as for the instructor. The instructor needs windows on the learners' conceptions and learners need ways in which they can confront their own deeply engrained conceptions.

2.1. *Re-learning for Pre-service Elementary School Teachers*

Prospective teachers are often in a position of having to re-learn specific elements of their mathematics. However, they are frequently content with what may be superficial understandings: once they have formalised a procedure, it is difficult to re-visit the underlying concepts for deeper understanding (Hiebert and Carpenter, 1992; Wilson and Goldenberg, 1998). Moreover, the existence of prior knowledge, which is robust while also being incomplete and superficial, can result in significant difficulties. Markovits and Sowder (1990), for example, reported that pre-service teachers participating in their research were more successful in acquiring knowledge in a new mathematical content, than re-learning what is perceived as 'familiar.'

One such 'familiar' topic is that of rational numbers. Through their own schooling, prospective teachers learn many procedures associated with converting numbers from one representation to another and operating on different types of representations. As such, many feel that they know fractions or know decimals – these are 'familiar' entities to them – and are thus reluctant to re-visit some underlying concepts associated with rational numbers. Yet student understanding of rational numbers is very complex, as the large body of research literature studying it attests to (Behr et al., 1984; Lamon, 1999). It is therefore not surprising that, despite their reluctance, pre-service teachers do not have a good grasp of rational numbers (Post et al., 1988).

One of the clearest findings of research on learning fractions and rational numbers is that of the many different ways in which fractions can be understood (as part of a whole, ratio, quotient, and number), students who come to understand fractions as numbers have the greatest success and flexibility (Lamon, 2001). Yet developing such an understanding has proved to be difficult, in part because many of the helpful visual representations used in the teaching of fractions tend to emphasise the part of a whole aspect of fractions, which is non-trivial to reconcile with the idea of fraction as number.

A primary goal was therefore to give our research participants the opportunity to re-visit fractions in a context that emphasises this 'fraction as number' way of understanding fractions, and that makes it possible for them to work with fractions and their decimal representations interchangeably. Given our previous research on various aspects of elementary number theory (see Campbell and Zazkis, 2002), we saw this as an opportunity to bring to the fore some related concepts, such as the difference between rational and irrational numbers. We wanted the pre-service teachers to encounter novel or unexpected properties and relationships such as the period of a fraction in concepts they had perceived to be entirely familiar both in order to increase the likelihood of prompting reconstruction and reorganisation and to challenge the participants' I-already-know-this complacency. Our research participants were both short-term and long-term re-learners, as they had learned rational numbers both long ago (in grade school) as well as quite recently (in a university-level mathematics course).

To satisfy both our research aims – to understand better some of the participants' conceptions about certain aspects of rational numbers – and our teaching objectives, one author (Sinclair) designed a computer-based learning environment focusing on the domain of rational numbers. We first outline our rationale for developing a rational numbers microworld. Then we describe the design principles we followed, and conduct a mathematical analysis of the concepts and behaviours supported by the microworld.

2.2. *Affordances of Computer-based Learning Environments*

Many researchers have argued, along with Goldenberg (1989), that well-designed computer-based learning environments can provide a scaffold for reasoning by fostering the development and use of visual and experimental reasoning styles, which greatly complement the traditionally taught symbolic deductive methods. For the learning of rational numbers, few such computer-based environments exist, particularly in comparison to the number that exist to support the learning of geometrical ideas, functions, probabilities and data analysis.

Pies, rectangles and number lines provide one type of visual representation frequently encountered in the sub-domain of fractions. Independently from these visual tools, calculators can support a certain range of numeric experimentation, particularly in terms of converting fractions to decimals. However, each tool is often used in isolation: neither pies nor number lines can be represented on calculators. In contrast to these isolated tools, there exists a class of environments called "microworlds" which, as Edwards (1995) describes, "embody" or "instantiate" some sub-domain of mathematics. The *microworld* is intended to be a mini-domain of mathematics that essentially brings such tools together into a phenomenological whole. Thus microworlds can be seen as specific forms of external representation of a subset of mathematical ideas. The challenge for mathematics educators is to design microworlds that can offer new external systems of representation that foster more effective learning and problem solving.

Noss and Hoyles (1996) argue that the computational objects of a microworld – its basic building blocks – should maximise the chance to forge links with mathematical objects and relationships. In the case of rational numbers, the computational objects should maximise the learner's chance to explore relationships among different types of numbers (rational, irrational, terminating, repeating, etc.), as well as their underlying structure. In terms of design, Noss and Hoyles also argue that the development of a microworld should involve predicting where student breakdowns might occur: breakdowns are incidents where learners' anticipated outcomes are not experienced. Microworlds that adhere to such design criteria provide the learner with opportunities to build new meanings through new external representations. One final design issue we wanted to bear in mind relates to the difficulties involved in incorporating visualisation into mathematical activity. We recognise that without proper support, students are either unwilling or unable to take advantage of potentially powerful visualisations (see Eisenberg and Dreyfus, 1991). We also recognise that microworlds are essentially inert without the animating presence of both the student and the task that invites investigation. The design of our learning environment thus included the creation of the microworld as well as a set of accompanying tasks that would encourage connections between analytic and visual modes of reasoning (see Appendix 1). Many of these tasks involved the comparison or simultaneous use of different denominators and numerators while others were intended to target particular properties of certain denominators.

In the next section, we describe a computer microworld – the Colour Calculator – and draw attention to some of its features, particularly as they compare to traditional paper-and-pencil environments or handheld calculators. This will give a sense of the kinds of properties and relationships that explorations with the Colour Calculator can emphasise, or make more transparent. We then describe a study with a group of pre-service elementary teachers, who used the Colour Calculator as part of their mathematics course. Finally, we describe the types of meanings that different participants were able to construct, and the ways in which their orientations (including attitudes and emotions, as well as richness of understanding) toward fractions and decimals were affected.

3. FRACTIONS AND DECIMALS WITH THE COLOUR CALCULATOR

3.1. *Description of the Colour Calculator*

The Colour Calculator is an internet-based[1] calculator that provides numerical results, but that also offers its results in a colour-coded table. Conventional operations are provided, as shown in Figure 1. Each digit of the result corresponds to one of 10 distinctly coloured swatches – reflected in a legend – in the table.

The calculator operates at a maximum precision of 100 decimals digits, and thus each result is simultaneously represented by a (long) decimal string and a table, or grid of colour swatches. It is possible to change the dimension, or the width, of colour table to values between 1 and 30. Figure 2 shows the result of typing 1/7 into the calculator with the grid width set at 10, thus generating the associated coloured table.

Using the button that controls the width of the table of colours, different table dimensions can be selected, which results in different colour patterns. Figure 3 shows 1/7 displayed using a table width of 18 and 17, respectively. The table width of eighteen produces a pattern we call "stripes" while the table width of seventeen produces a pattern we call "diagonals."

Of course, because of the way numbers are displayed in the Colour Calculator – only the digits after the decimal point are represented in the coloured table – the division operation produces the most interesting results, particularly when the rational quotient has a repeating pattern. In fact, the Colour Calculator has the effect of reversing

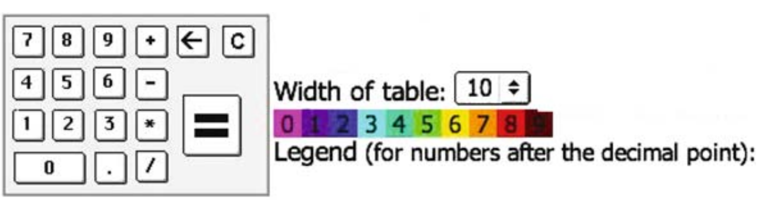


Figure 1. The Colour Calculator.

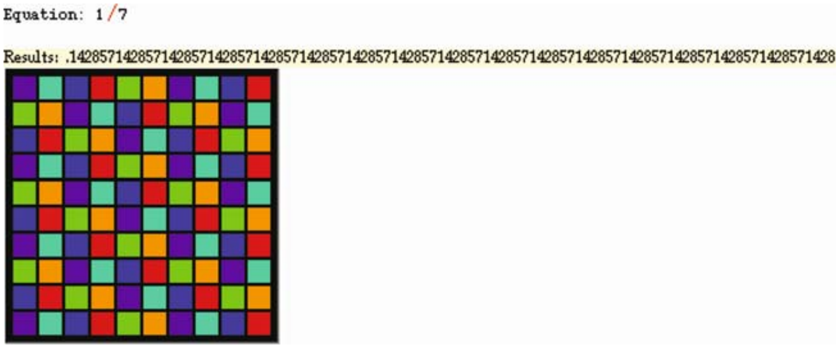


Figure 2. The Colour Calculator showing 1/7.

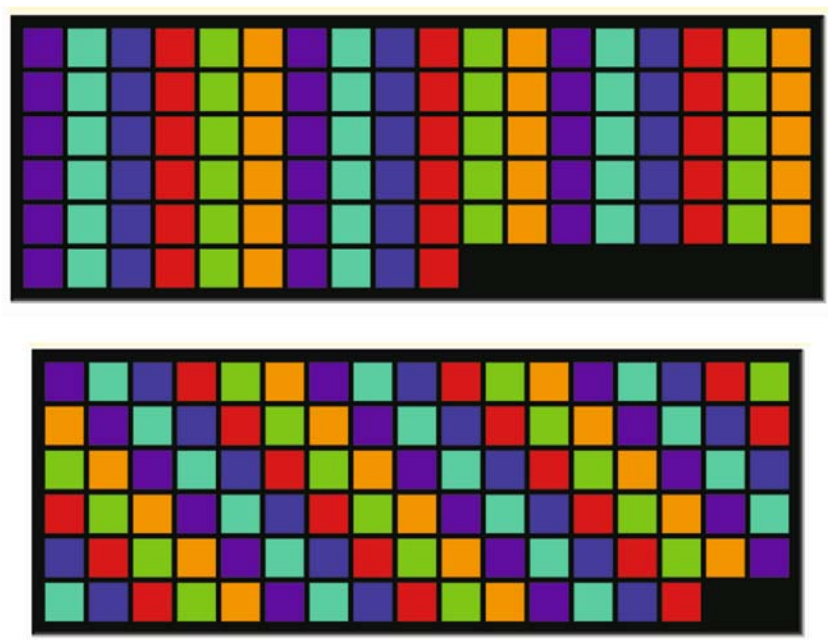


Figure 3. Different representations of 1/7 in the Colour Calculator.

previously 'transparent' and 'opaque' (see Zazkis and Gadowsky, 2001) properties of rational numbers. For example, the representation 1/7 makes *transparent* the number's part-wholeness, while making more *opaque* the fact that its decimal representation is non-terminating decimal. In contrast, the representations of 1/7 shown above make *transparent* the number's periodicity, while making more *opaque* the fact that it corresponds to a simple fraction.[2]

3.2. *Faster than a Pencil, Bigger than a Calculator, and More Colourful than a Textbook*

We see the power of the Colour Calculator as an instructional tool as being rooted in three important features that are not found either on the handheld calculator or in paper-and-pencil environments: colour, speed, and size. The size feature relates to the number of digits that are calculated and displayed (100 instead of the typical handheld calculator's 8). With only eight digits, the handheld calculator can barely display the repeating pattern found in a fraction with a denominator as little as seven (since its repeating pattern is of length six, the rounded eighth digit can easily obfuscate the pattern). In contrast, the Colour Calculator easily handles much larger units of repeat. This means that repeating decimals repeat transparently, and therefore stand in greater contrast to non-repeating decimals. We hypothesised that the number of digits displayed can help learners create a perception that infinitely repeating numbers really *do* go on and on, and that non-repeating infinite decimals really do *not* repeat.

The speed feature relates to the Colour Calculator's ability to compute and display quickly the decimal expansion of a fraction; this contrasts especially with paper-and-pencil environments where the process of conversion from fraction to decimal is frequently long, tedious and error-prone. Speed thus allows learners to see a much wider range and greater number of decimal expansions (it makes no distinction between 'hard' fractions like 3/41 and 'easy' fractions such as 5/6), and to test conjectures during exploration and problem solving more quickly. In fact, it draws attention away from the procedure of conversion to the properties and relationships that exist between common fractions and their decimal representations. This should allow learners to work with the results of the conversion and to treat these results as objects rather than processes.

Finally, the colour feature is responsible for translating strings of digits into a format where patterns are more easily discernible. Since the colours are displayed within a manipulable grid, they can be seen all-at-once (compare the tabular representation with the one-dimensional array) and can be flexibly arranged to reveal certain patterns. (The grid width of 18 in Figure 3 arguably provides a more revealing pattern than the grid width of 10 in Figure 2.) Many of these patterns are recognisable and attractive (stripes, diagonals, and checkerboards – see Appendix 1 for an illustration of these three types of patterns) and can thus become motivational objects: Can you create a grid of stripes? Can you create an ''all-red'' fraction? These kinds of question are qualitatively different from: Can you find a fraction that has a repeating decimal representation? First, they involve creating a certain object, and, second, they draw attention to the actual value of the length of the repeating decimal's period.

The three features of the Colour Calculator are intended to encourage and support experimentation. Unlike other learning environments involving rational numbers, this one makes it feasible to investigate the relationships that exist between, say, the denominator of a fraction and the period of its decimal expansion. Since particularly interesting relationships occur with denominators that are prime or ones that are of the form 10^n-1 (so 9, 99, 999, etc.), the tasks we designed were oriented toward familarising the participants with those relationships. In spite of its speed, size, and colour, the Colour Calculator can also provide an exploratory environment well-suited to concepts previously explored in slower, smaller and less colourful contexts. For instance, while the Colour Calculator is especially powerful for displaying infinitely repeating decimals, it can also display terminating ones, and in fact, it provides a starker contrast between the two than is possible to achieve with other tools.

4. THE RESEARCH STUDY: A WINDOW ON RE-LEARNING

4.1. *Setting*

Participants in this study were pre-service elementary school teachers enrolled in a course *Principles of Mathematics for teachers*, which is a core one-semester course in the teacher certification program. The course is intended to deepen and extend students' understanding of topics underlying the elementary school mathematics curriculum. The Colour Calculator was one of two ''project''-options participants had to choose from as one of the requirements of this course (the other option involved an investigation that was not computer-based). Out of 90 students enrolled in the course, 42 chose to work with Colour Calculator. Out of these 42, 7 volunteered to participate in a clinical interview related to their experiences with this microworld.

Participants' work in the course included chapters on Fractions and Decimals (Musser et al., 2003). These chapters included the topics of decimal representation of rational numbers, conversion between common and decimal representation of fractions, classification of repeating decimals, representations of rational vs. irrational numbers, among others. These topics were completed shortly before the participants were introduced to the Colour Calculator.

The participants were provided with written instructions about web access to the Colour Calculator, a description of its commands and capabilities, and a list of suggestions for mathematical explorations to be carried out before turning to the main assignment. These tasks were intended to familiarise the participants with all the aspects of the Colour Calculator and create an environment of experimentation and conjecturing. To support the participants, computer lab hours were scheduled at different times on different days. Participation in the lab was optional; nearly all the participants (and all the interview participants) chose to work, at least for part of the assignment, in this environment.

Our data consists of two main sources. The first source includes observations of the participants' work during the lab time, when they worked on series of tasks assigned to them by the course instructor (see Appendix 1). During the allocated lab hours, the work of the students was observed and supported where necessary. We noted frequently asked questions, chosen routes for exploration, student's conjectures, as well as their approaches toward testing their conjectures. We also used these observations as a guideline for designing the interview questions, which constitute our second source of data.

Our lab observations were focussed primarily on identifying instances in which the participants engaged with concepts or ideas they had already encountered, either in their university course or in the previous schooling. Thus, we initiated interactions with participants who communicated various forms of recognition. For instance, if a participant noted a similarity with something they had encountered in their university course or mentioned having heard of a certain idea sometime during their pre-university schooling, we followed-up with requests for clarification. We also followed-up with participants who expressed any form of surprise, in order to probe the understandings they were bringing to their work with the Colour Calculator and to discern how these understandings were being challenged.

Within 2 weeks of the completion of the written assignment, clinical interviews were conducted with seven volunteers from the group, who represented a range of abilities in the group. The interviews were audio-taped and transcribed. The interviews lasted

30–50 minutes; the Colour Calculator was available to the participants at all times during the interview. These interviews had a semi-structured character, that is, the questions were designed in advance (see Appendix 2), but the interviewer had the liberty to follow-up with prompts, include additional questions, or omit questions due to time considerations. In our interviews, we posed a series of questions that were designed to assess various components of the participants' understanding and to further probe the kinds of cognitive, affect and aesthetic representations that the participants had constructed in their interactions with the Colour Calculator.

In addition to these two sources of data, we also drew supporting evidence from the participants' written work from their projects, particularly the descriptions of their visual images of fractions and decimals elicited in the Final Task. We begin by reporting the recognition instances – involving familiar though contextually different ideas for the participants – observed first during the lab observations, and often re-visited during the interviews. We then discuss the cognitive effects of the participants' use of the Colour Calculator on the problem-solving situations given to them during the interviews, followed by some affective and aesthetic effects.

4.2. *Lab Observations: Enriching Formal Understandings*

We had a total of 42 participants and most stayed in the lab for at least 2 hours. Some of these participants worked in pairs or groups at one computer while others worked individually, though usually sharing findings and observations with their peers. In what follows, we have chosen to describe several of the recognition instances that occurred most frequently in our observations. When appropriate we have drawn on the interview data in order to further illuminate or support our observations.

4.2.1. *Magic 9's*

During their university course, the participants had been exposed to the algorithm for converting decimal numbers into fractions. They had completed an assignment in which they had been asked to use this algorithm for a number of decimal numbers. Task 2 (see Appendix 1) asked the students to investigate the several fractions whose denominators are of the form 10^n-1 (such as 9, 99, 999, etc.). We observed that the participants were very surprised to find that, for example, 24/99 yields a repeating decimal with unit of repeat 24.

However, they easily worked through the questions in this task, generalising the "magic 9's" pattern and noting some special cases such as 24/999, where the repeating decimal is not $0.\overline{24}$ but instead $0.\overline{024}$. It is when they began encountering these exceptions that they started to connect the phenomenon of the magic 9's to the algorithm learned in class (exemplified below). Although they could use the algorithm successfully, which takes a decimal number and turns it into a fraction, they had not connected the output of the algorithm *back* to the decimal number; the algorithm was a one-way street.

$$\text{Let } n = 0.\overline{238}$$

$$\text{Then } 1000n = 238.\overline{238}$$

$$\text{So } 1000n - n = 999n = 238$$

$$\text{And } n = 238/999$$

One pair of participants remarked that their work with the Colour Calculator made them realise what the algorithm was actually doing and how it was connected with fractions. Though the algorithm always features nines (when the repetition starts immediately after the decimal point; see the third and fourth steps above), those nines seem to remain somewhat opaque for students, who focus on the algebraic manipulations involved. We observed many groups of students writing out the algorithm on a piece of paper, accompanied by exclamations such as "Oh! Now I see what's going on." Some seemed to be referring to what they were seeing on the screen while others were referring to the algorithm. At any rate, a connection had been made for them. Andrew in particular commented on how working with the 9's in the Colour Calculator made "theories that were out there become related." In an interview, Blake also commented on the relationship between what he learned in class and what he learned while working with the Colour Calculator: "I think that really like using the numbers over 9, that really like related that, because it's, you never really work backwards with that formula, or whatever, but in this case you are forced to go the other way, which really makes it make sense. You know, it just gives you a better understanding of that, that it really works." This interplay between the "theory" learned in class and the "practise" was also apparent in Leah's interview: "Um, I think just better understanding, because I remember before, Peter had mentioned something about, you know, there's a very abstract theory for it, but then seeing it and using it in practice really related it.?"

Perhaps students tend to view the algorithm as a process – or even a trick – that allows them to turn decimal numbers into fractions rather than as a general relationship between fractions and repeating decimals. We hypothesise that the patterned display of the Colour Calculator helped that relationship become both more apparent and interesting for our participants. It gave the participants an invitation to "play," as Blake described later in an interview: "No, I didn't know that at all, because, well like I, you know, I kind of knew them but I just, I didn't relate the two, because I hadn't seen it like that, we didn't play with any numbers like that, so I just didn't make the connection." And it gave them the opportunity to "see," as Leah later described: "[the instructor] talked about it in class and then yeah and then I did it on here just to see what it looked like, that was weird. This was a more visual thing, it was, I mean it's the same thing, it's just, just thinking about it more." As we will discuss later, the participants also gained an appreciation of how the "magic 9's" phenomenon yielded a process that allowed them to generate any repeating decimal they wanted, thus reversing the direction of the algorithm.

4.2.2. *Different but Equal*

As we observed the participants working in the lab, we were surprised by the number of times we heard comments about the fact that the fraction they had inputted and the decimal string that was being represented in the table were somehow *one and the same*. This realisation would typically occur at the beginning of their sessions with the Colour Calculator, and was evidenced by comments similar to those made by the middle school students reported in Sinclair (2001): "oh, those are the same" or "I've never seen them together."

Apparently, fractions and decimals *look* very different and it therefore requires some cognitive work to see them as both representing the same number. Prior to using the Colour Calculator, the participants were perhaps more influenced by the *form* of these different representations than by the quantities they denote. Typically, students learn algorithms for converting one form into the other, but the assertion that '1/4 can be represented as 0.25' is not the same thing as '1/4 is equal to 0.25.' Moreover, since decimal representations of fractions are frequently rounded, they are actually only approximations ($1/3 \approx 0.33$). This may obscure students' appreciation of the fact that fractions *are* numbers, and are thus equal to their decimal representations – which students are more likely to perceive as numbers. In the Colour Calculator, even though a repeating decimal is ultimately cut-off, and thus is an approximation of the fraction, the participants were able to interpret the patterned image of the repeating decimal as an infinitely repeating *object*, and thus as equal to the fraction.

4.2.3. *The Beauty of the Decimal System*

When the participants were first working with the Colour Calculator, trying out different fractions, they frequently expressed some form of disappointment when they had generated a terminating decimal. This was often with fractions such as 1/4 or 3/10. We sometimes chided them, asking "but don't you *know* that 1/4 gives you 0.25?" We also reminded them of the criterion they had seen in their course that checks whether or not a fraction terminates (can the denominator of the reduced form be written as 2^n5^m?). Some of the participants insisted that they did in fact know, but that "I just wanted to see" or "it's so easy to just punch it into the calculator without thinking."

Although we suspect that many of these participants would have been able to determine correctly that 1/4 and 3/10 terminate, we would argue that their purported knowledge was still rather vulnerable. Moreover, we observed at least three groups who groped their way toward the criterion by first focussing on whether the denominator was a factor of 10, 100, or 1000. This may be due to the ease with which they could experiment. By trying 3/8 and looking at the decimal representation, it became obvious to them that there are three decimals and that 8 is a factor of 1000. Two different groups of participants expressed surprise at this relationship between the way decimal numbers are written (in the decimal system) and the denominators that generate terminating fractions. What role does the Colour Calculator play in prompting this realisation? First, it simply allows students to test many fractions and quickly gain feedback. Second, it is easier to distinguish a much larger class of terminating fractions than it is in paper-and-pencil or regular calculator environments: a fraction as simple as 1/16 may not appear as obviously terminating on a regular calculator. Third, we suggest that students need to encounter similar ideas in a variety of settings. Their recollection of the algorithm likely contributed to their developing experience with a large range of terminating decimals. And the novelty of the setting may have motivated the participants to experiment rather than to assume they have already "covered" the ideas involved. Many participants, such as Christine, were able to admit that her experience helped her "re-establish things that I had forgotten about decimals or about fractions."

4.3. *Interview Findings: The Transparencies of the Colour Calculator*

In this section we report some of our findings from the semi-structured interviews. The interview questions were designed to probe the participants' understanding and familiarity with terminating, infinitely repeating and infinitely non-repeating decimals.

4.3.1. *Generating Fractions from Patterns of Decimals*

During the interviews, every single participant was immediately able to propose examples of fractions that would generate certain patterns (see Question 5 in Appendix 2).

When asked about the image shown in Figure 4, four of the participants responded as Andrew does here, recalling specific fractions from his lab work (after he identifies the denominator needed, he must identify the grid width, which he calls "the pattern," that will produce the diagonal pattern):

Andrew: (pause) purple, (pause) 4, 2, it's a pattern repeated, 6, so from previous experience I know that 7 and 13 all have a repeating pattern of 6, um I'm going to go with (pause), I'm going to try 7...

Interviewer: Okay, so 1/7...

Andrew: (pause) purple, red, orange, that's the one I want, and we set the pattern at 11.

Interviewer: 11, how did you know 11?

Andrew: Um, yeah, okay, because you've got the repeating pattern of 6 and then you basically yank the last one and start the next line, or would you bump it up or.

The other three interviewees relied on the "magic 9's" and proceeded as Darren did below:

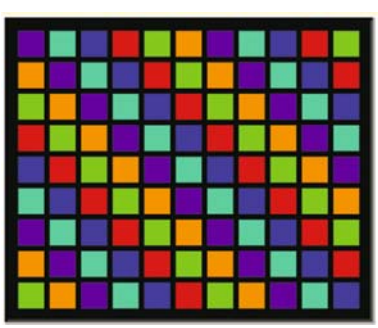


Figure 4. Colour Calculator pattern used in the interviews.

Darren: I'm just going to see if there's a pattern of repeat, I think there is a pattern of repeat, so, (pause) okay so the pattern of repeat is 6, the digits are ...

Interviewer: It doesn't matter if you don't get exactly the colours, as long as you get the pattern. So you got here what, 1, 4, 2, 8, 5, 7, okay...

Darren: Yeah, so I have a repeat of 142857, now I'm going to, the denominator will be rather large, it will be about 99,000...

Interviewer: Okay, try it.

Darren: Yeah it'll be, okay I'll try that, (pause) ...

Interviewer: 142857 and how many 9's are you going to put...

Darren: I'm going to put in, I'm going to put in six 9's.

With those who used Darren's method, the interviewer asked whether there was a smaller fraction that would work. One participant was able to identify immediately a fraction with a denominator of seven while the other two proceeded by simplifying the 9's fraction they had generated. During the interview, with the prompting of the interviewer, these two participants were able to see that sevenths, for example, had a period length of because they could be written as fractions over 999,999. Nicole remarked that "13 was a 6 pattern, so it would obviously go into 999,999 some amount of times, I guess there we go, that's why it works. So there, I just learned something new, because of the colour calculator."

For the third image shown in Figure 5, every participant used the 9's method and found that 226/999 generated the correct pattern. And for the fourth image (see Figure 6a), one participant attempted to use the 9's method, after having correctly determined the period length of 46[3], while the others were able to identify immediately 47 as the correct denominator. In addition to being able to call upon 'basic' fractions such as 1/3, 1/4, 6/10, 2/5, the participants knew what fractions having denominators such as 7, 13, 17, or 47 would look like (one participant commented that "Well I know, I know I can now say well 7 is actually a routine example with a length of 6"), and they could describe some relationships between the denominators and the decimal expansions (particularly for prime number denominators and denominators of the form 10^n-1). This kind of relational understanding – which differs from the participants' existing algorithmic way of understanding – might account for Danielle's claim "I now have a better understanding of how certain fractions create different types of decimals, such as finite or infinitely

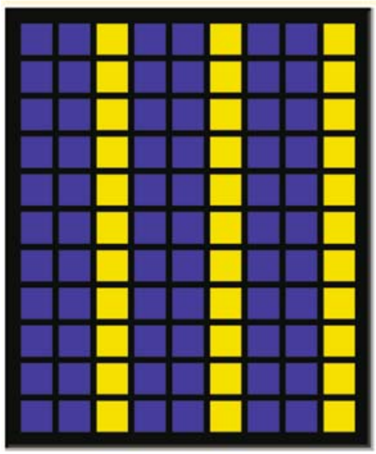


Figure 5. Colour Calculator pattern used in the interviews.

repeating." She may not have had a better understanding of how to convert fractions to their decimal representations, but she had a larger set of concrete (and colourful) examples corresponding to different types of decimal representations.

It is interesting to note that students are frequently more comfortable with 'nice' fractions that terminate than with fractions that repeat infinitely. Students' typically biased exposure to these 'nice' terminating decimal representations of fractions may mislead them into believing that fractions having terminating decimal representations are the most common types of fractions. Referring to the frequency of occurrences of non-terminating fractions relative to terminating ones, Blake noted that he *now* thought that terminating fractions were "weird compared to what we've been doing with this

(a) 3/47 on the Colour Calculator **(b)** $\sqrt{2}$ on the Colour Calculator

Figure 6. Two outputs from the Colour Calculator set at grid width 9.

program." An increased exposure to non-'nice' fractions acquainted Blake with the preponderance of non-terminating numbers. His comment reveals the way in which students' typical experiences with rational numbers may bias them towards thinking that terminating numbers are more frequent than non-terminating ones. It may be that just having access to a larger set of concrete examples of non-terminating decimals can rectify this bias. Further, working with the Colour Calculator may also help students develop associations with non-terminating numbers, the way they might have with terminating decimals such as 0.25 (they know its associated fraction, they have encountered it in the context of currency).

4.3.2. *Focus on Denominators: Repeating Versus Non-repeating*
In addition to gaining more familiarity with the difference between terminating and repeating decimals, participants worked with several examples of irrational numbers in the lab, that is, numbers that have infinite non-repeating decimal representations. These types of numbers are frequently difficult to discern with handheld calculators, which only display eight digits. Christine explained how calculators had left her confused about the status of many decimal representation: "I think it helps a lot that is shows so many numbers, like on a calculator you would just see, like if you were to take 1/47, it would look like it's an irrational number." The Colour Calculator helped the participants realise that fractions such as 1/47 can have very long periods (in this case, 46), yet still repeat. Christine knew the definition of irrational numbers but had been unable to reconcile this definition with the results shown on regular calculators. However, by building up a strong image of many, many repeating decimals, she could more distinctly grasp the notion of a non-repeating infinite decimal. This became very clear in the interviews when the participants were asked to explain how certain coloured patterns had been created with the Colour Calculator. Even an image such as in Figure 6a posed no problems, for the participants knew that some fractions could have long periods. In this excerpt, Christine shows that she simply needs to find whether the first sequence of digits (or colours) repeats again:

Interviewer: Okay, how could the Colour Calculator produce this image?

Christine: Umm first I'd find some colours that repeats, so the starting goes pink, yellow, blue, red, so I need to find that sequence somewhere, what's yellow, blue, red, so it would repeat, 1,2,3,4,5,6,7,8,9, 1,2,3,4,5, so 45, 46 times, so the denominator would have to be 47, because that's a prime, so it'll give 46 repeaters.

Instead of restricting themselves to the first digits in the sequence as Christine does, most interviewed participants were drawn to a certain local pattern that occurred twice in the image, thus recognising that a repeating pattern in the decimal representation will create a certain pattern within the table of colours. For example, the diagonally-arranged powder blue squares pop out for Andrew, and enable him to assert that there is indeed a pattern, and that its length is 46.

Andrew: This is an awfully long pattern.

Interviewer: What do you see...

Andrew: Well I see those two blue guys right there. They're repeating further down there and I'm sure there is a pattern and then once you get these blues here you can count 9, uh, 18, 27, 36, this is an awfully long pattern, 36, oh 45, 46, so it's probably something over 47.

With the image in Figure 6b, most of the participants reasoned, like Leah below, that they could only deduce that either the period was greater than the number of colours shown, or the pattern represented an irrational number.

Leah: I'm just trying to see if I can see a pattern at all. Yellow here, and like there's one, that yellow there. No, it doesn't repeat, you never get these two guys, these three guys here, it looks like irrational number.

Interviewer: You say it looks like an irrational number, why?

Leah: It looks like, because I can't find any pattern, I mean it could just be that this is, the pattern goes from here to here.

Leah's interview shows that she has clearly understood what a non-repeating decimal will look like, and has even grasped the limitations of the Colour Calculator's output. Darren placed somewhat more trust in the Colour Calculator in his description of π: "I realize that it can't be a repeating decimal and so, I can say, I could go into the colour calculator and it can show me what pi really looks like as a colour."

4.3.3. *Some Special Denominators*

We now consider a typically less familiar relationship than the ones described above. Students frequently learn how to convert a fraction to a decimal using long division, but this process does not make explicit the role played by either the numerator or the denominator in determining a fraction's decimal expansion. Through their use of the Colour Calculator, we observed that several participants were just beginning to realise that the denominator of a fraction plays an important part in determining the decimal expansion. This realisation has familiar as well as novel components, and Nicole describes them both: "Regarding fractions, my visual image depends a lot upon the denominator, which plays a large factor in determining what type of decimal representation [terminating or repeating] will occur as well as how long the period length will be." The participants were previously aware that different fractions produced different decimals but the specific role of the denominator – which determines the length of the period – became more pronounced, as Leah points out: "I didn't know anything about like denominators having anything to do with the length of the periods and stuff."

Not only does the denominator play a large part in determining the pattern of a decimal expansion, but there is also a relationship between the denominator and the period length. Kyle's articulation of this property – which is not completely accurate – is typical of the understanding achieved by participants: "specific denominators will consistently have a certain period length no matter what the numerator is." This understanding is elaborated upon by Tracy, who sums up the role of both parts of the fraction: "The numerator is responsible for determining which numbers are in the decimal but the denominator dictates the period length and therefore is responsible for the structure of the decimal."

4.3.4. *Focus on Numerators*

Prior to their work with the Colour Calculator, the participants had been exposed to (and practised, through a homework assignment the previous week) the relationships between different fractions with the same denominator. The homework assignment involved looking at the decimal numbers for the family of sevenths (1/7, 2/7, 3/7, 4/7, 5/7, 6/7). The participants were also asked to do this with the Colour Calculator. We had thought this would be trivial for them, a mere repetition of their homework. Yet, several participants still found their results surprising. Andrew explained the source of his own surprise as follows: "That was surprising to me, you don't even see that when you do it on the calculator and your calculator has a tendency to round for you, so you never see that 1/7 has same numbers as 2/7." Granted, this property – that each member of the family has the same sequence of numbers, though starting at a different term in the sequence – lays outside the realm of material usually covered in school mathematics, but Andrew explained how it added to his previously flat understanding of infinite decimals:

> I don't think I would have known that because my previous experience with any decimal, I mean in high school they just said, that's a never-ending decimals, 1/7 is a never ending decimal. That's what we were told. Just put three dots beside it and don't worry about it. Well, no more little dots for students of mine, no way.

The use of the "three dots" ended up obscuring many relationships for Andrew. By knowing more about the family of sevenths, including the length of its period and the relationship between the numerators and the actual numbers in the period, the fraction 1/7 gained some personality.

4.4. *How were the Participants' Orientations towards Fractions and Decimals Affected?*

As the participants began working with the Colour Calculator in the lab, they were engaged and explorative. Many responded with surprise and even delight upon creating their first patterns of stripes and diagonals. Both in their written assignments and during their interviews, the students had opportunities to describe qualitatively their experiences working with the Colour Calculator. They were explicitly invited to do so in the Final Task of the assignment and in the last interview question. Two major themes emerged: vivifying properties and relationships encountered more formally in the classroom and developing understanding through experimentation and visualisation.

4.4.1. *Vivifying Properties and Relationships*

Colour seemed to play a role in vivifying the participants' understanding of various properties and relationships. It is difficult to tell whether she could have described 1/3 prior to using the Colour Calculator, but Kimberley wrote in her project that a "denominator 3 will always give me a block of mono-colour in my mind." She adds "I'll forever see fractions and decimals in colour." In a slightly different vein, Darren talks about how the colour-coded decimal representation helps make clear distinctions that might otherwise be vague: "Fractions in their decimal representations (with colour) give a better understanding of the relationship between fractions and decimals. Thus 3/99 and 3/9 seem similar but the results are very different."

Some participants described and wrote about the way in which the Colour Calculator helped bring to life concepts they had encountered in the classroom. Christine talked about the Colour Calculator gave her shortcuts to doing things: "Like with this whole program, the questions we would try to figure them out how we learned them in a lecture, or just how we would figure them out, and then we'd realise, there's a shortcut to it." Once again, Christine is describing how the Colour Calculator helped make certain ideas more accessible, more flexible to work with. In her statement, Kelly alludes not only to the vivification of the logical procedures encountered in the classroom, but also to the role that the actual tasks played in her experience: " I have been able to gain a whole new perspective of fractions and decimals. These tasks forced us to look for logical procedures as opposed to stumbling upon the correct answers. Looking to create stripes, diagonals, and checkerboard patterns on the grid allowed me to search for an understanding rather than being handed the answers."

Part of the process of vivification can include developing emotional or aesthetic responses to mathematical ideas. Aimee clearly was able to do both using the Colour Calculator, as she reports in her project:

> The repeating fractions have a flow that I find comforting. As a child I disliked fractions that did not terminate, but now I see them in a light of beauty. I find that the decimals which terminate sad. They are unable to touch the fingertips of forever, like the repeating ones can.

Though they did not make statements such as Aimee's, we noticed that many participants had strong aesthetic responses in the lab (admiring the patterns created, commenting on the appeal of the display, expressing surprise and pleasure). In their projects, several participants used words such as "appealing," "pretty," "nice" and "aesthetic" in their descriptions. We also saw strong aesthetic responses during the interviews, as we will elaborate shortly. Some of them even developed strong preferences for certain types of patterns over others, such as stripes over diagonals; these preferences emerged when they were asked for the best way of displaying different fractions.

4.4.2. *Developing Understanding through Visualisation and Experimentation*

As Kelly's comment above states, the assignment tasks encouraged experimentation. In fact, we propose that both the use of colour representation and the possibility of experimentation were crucial to the participants' experiences with the Colour Calculator. With regard to experimentation, Danielle notes that "with this program you can try several things and just play around with fractions and decimals until you figure out patterns and simpler ways of finding what you want." Because of this, some of the participants were able to engage in exploration and problem solving, which for Brad, "helped me justify some unknown curiosities."

With regard to the visual representation afforded by the Colour Calculator, Kevin wrote about the increase in pattern possibilities now available: "patterns can be seen as you move left to right, up and down, and even left and up, or right and down. Patterns are more visible and meaningful in the sense than when written down on paper in the standard 0.blah blah blah blah blah blah way." A few participants talked about the way in which the colourful patterns attracted and held their attention, in a way that numbers might not. For example, Kyle noted that "without colours to represent numbers, patterns are much more difficult to discern, and can impact highly on an individual's ability to focus. Similarly, Kelly explained "the colours were important in not only clarification purposes, but were also important in keeping me attentive." The colour also seems to make patterns more accessible than they are on handheld calculators, as Dianna insisted: "Being able to see the numbers represented as colour helps the patterns to become more pronounced for me. Normally, on the regular calculator, you cannot see that there are sets of repeating numbers – they usually just look like a jumble with no rhyme or reason." Lauren agreed: "It is easier to see patterns of colours than patterns of numbers."

5. CONCLUDING REMARKS

Frequently, pre-service elementary school teachers do not have a deep understanding of the mathematical topics related to the school curriculum. However, they do possess a superficial understanding, a certain familiarity with the mathematical concepts in question, and thus need to embark on a process of re-learning. But re-learning does not necessarily come from re-teaching, a process that can be contaminated by existing understandings, inaccurate or fragmented as they may be. A construction metaphor is illustrative: when an old house isn't solid, it is at times preferable to flatten it and construct a new one on the same lot, rather than strengthen, restore, and renovate the old structure. In teacher education, we do not have this option: we must work with a learner's existing structure, and we must reengage the learners in the process.

This is not an easy task in that students tend to shy away from reengaging with a concept that they already have some familiarity with, often overestimating their abilities and understandings of the concept in question. Furthermore, if the concepts are at all unpleasant, as 'fractions' and 'decimals' have a tendency to be for learners, then reengagement is even more challenging. In the specific case of pre-service elementary school teachers, deepening prior understandings may require new, or more sophisticated approaches than simply building new understandings does. In the context of education and relearning, such sophistication takes the form of experiences and challenges that provoke learners to re-examine and enrich their understandings of the mathematical concepts previously encountered.

In this study we have shown how pre-service teachers' interactions with a web-based colour calculator both enriched their experiences with rational numbers and challenged their understandings of several properties related to fractions and their decimal representations. The novelty and aesthetic appeal of the Colour Calculator – in synchronisation with the tasks we chose – helped the participants overcome their reluctance to reengage with properties and relationships associated with these concepts. Through colour, speed, and size of display, previously opaque qualities such as repetition and length of the period became transparent, and thus available for exploration and problem solving. More importantly perhaps, our research participants had the opportunity to engage with representations of fractions in decimal form as *objects*, rather than as final steps in a procedure, and to construct vivid, memorable, and positive images of important mathematical ideas.

NOTES

[1] It can be accessed on the web at: http://www.math.msu.edu/~nathsinc/ (follow the Alive Maths link). The design of this calculator was inspired by work at the Centre for Experimental and Constructive Mathematics (CECM) at Simon Fraser University, where techniques are being developed to employ the natural visual capacities of human perception to search for complex relationships and patterns in numerical distributions. Though the CECM's use of visual calculators is aimed primarily at looking for the fundamental underlying structures of mathematical objects such as sequences of polynomials and continued fraction expansions, author Sinclair designed a modified and

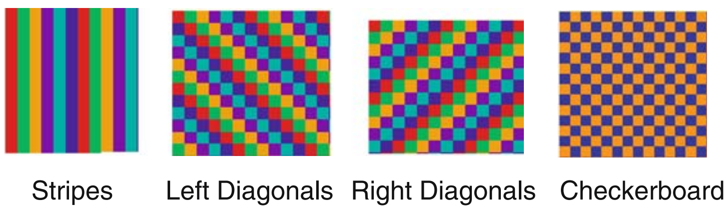
simplified calculator that would be more suitable for the exploration of simpler mathematical objects.

[2] In order to input a fraction, a user has to choose a numerator, then the division "/" symbol, and then the denominator. This series of actions certainly emphasises the notion of fraction as part-whole. However, as reported by Sinclair (2001), and as we show later, students seem to react to the notion that 1/7 *is* $0.\overline{142857}$ (as if they previously thought 1/7 was a computation that produced a certain decimal number).

[3] This would require inputting a fraction whose denominator is a 46-digit number, a computation that the Colour Calculator cannot handle.

APPENDIX 1

Written Response Tasks



Task 1: Stripes, Diagonals and Checkerboards.

1. Can you find three different ways to create stripes?
2. Can you find three different ways to create diagonals? What makes them right or left?
3. Can you find three different ways to create a grid with only one colour in it?
4. Can you find three different ways of creating a checkerboard pattern? What about a "real" checkerboard with a grid width of 8?

Task 2: Magic number 9.

1. Try the following fractions: 2/9, 5/9, 7/9. What do you notice about these fractions? Can you explain why the denominator of 9 has this effect?
2. Try the following fractions: 23/99, 48/99, 73/99. What do you notice about these fractions? Can you explain why the denominator of 99 has this effect? Can you create a grid that starts blue, orange, and repeats?
3. Predict what will happen if you display the fractions 123/99 and 145/999. Test your conjectures and explain what you observe.
4. Can you make a grid of blue and yellow repeating without using the digits 2 or 6 in your fraction?

Task 3: The period of primes.

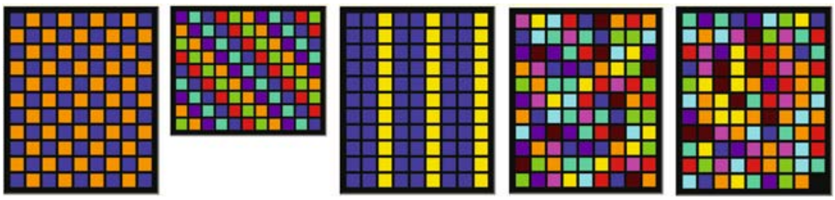
(a) Describe how you would set the width of the grid to best illustrate the period of each of the following fractions: 3/11, 1/29, 1/59.

(b) Display the fractions 1/7, 2/7, 3/7, 4/7, 5/7, and 6/7. What do you notice?
(c) For denominator ≤ 10, the longest period length is 6. What is the longest length of period you can find for denominator ≤ 0? Describe how you looked for/found it.
(d) Predict the length of period of the following fractions: 1/13, 1/17, 1/19. Can you predict the length of period of any fraction whose denominator is a prime number?

Final Task. Write a paragraph that describes the visual image you have of various fractions and decimals.

APPENDIX 2

Interview Questions



1. Can you get the fraction 11/13 into stripes?
2. How would you find a fraction that produces a table of stripes when the grid width is 9?
3. The grid only displays 100 decimals. Could you tell what the colour of the 103rd cell would be? What about the 175th cell?
4. We are looking at 14/99. Can you predict what will happen if we look at 140/990? Can you find another way of obtaining the same sequence of colours?
5. How could you make a table of colours that looks like each of these?
6. Can you use the Colour Calculator to show why π is not equal to 22/7?
7. What have you found surprising or helpful or interesting in your experience with the Colour Calculator?

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